Solutions Exam Signals and Systems

22 januari 2015, 9:00-12:00

Problem 1: signals and spectra
Each part is worth 4 points.

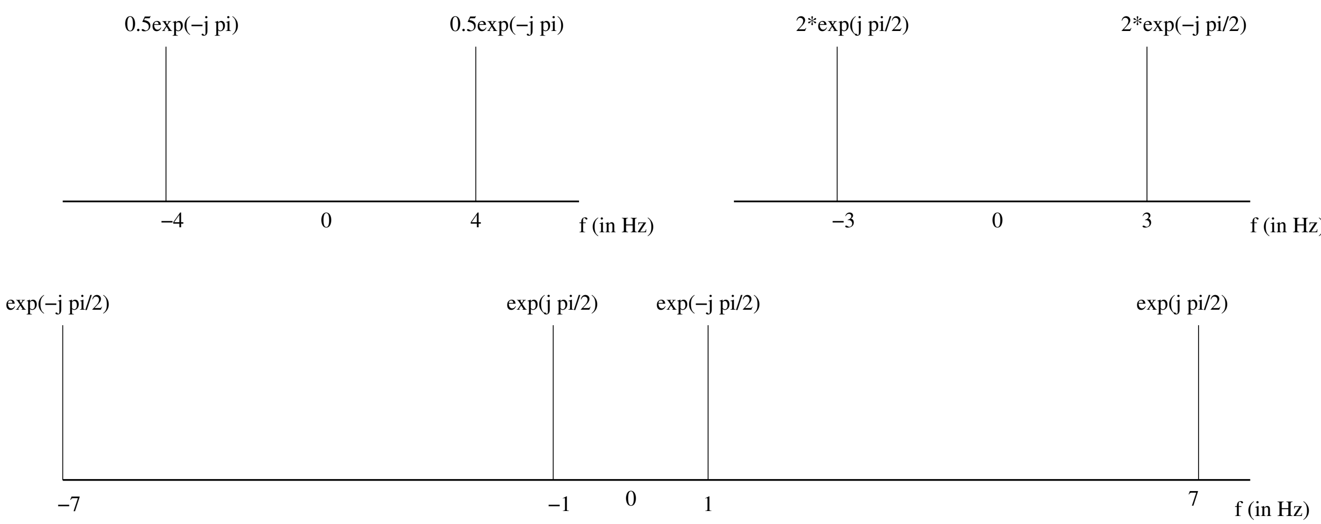
(a) The first signal is clearly a sine with amplitude 4 and is periodic after 2 seconds, so its frequency is 0.5Hz. Its delay relative to a standard cosine is 0.5 seconds, which corresponds with a phase angle of $-\pi/2$. We conclude $x(t) = 4\sin(\pi t) = 4\cos(\pi t - \pi/2)$. Similarly, the second signal is a cosine with amplitude 1, and frequency 5Hz, so $y(t) = \cos(10\pi t)$.

Careful inspection of the third plot shows that it is an AM signal that is constructed from the other two plots: $z(t) = x(t)y(t) = 4\cos(\pi t - \pi/2)\cos(10\pi t)$. Note, that we can rewrite this (using formula 9 of the formula sheet) into $z(t) = 2\cos(9\pi t + \pi/2) + 2\cos(11\pi t - \pi/2)$.

(b) We use the inverse Euler formula $cos(\theta) = \frac{e^{j\theta}+e^{-j\theta}}{2}$.

$$
\begin{aligned}
x(t) &= -\cos(8\pi t) = \cos(8\pi t - \pi) = \frac{e^{j\pi}e^{j\pi 8t}}{2} + \frac{e^{j\pi}e^{-j\pi 8t}}{2} \\
y(t) &= 4\sin(\pi 6t) = 4\cos(\pi 6t - \pi/2) = 2e^{-j\pi/2}e^{j\pi 6t} + 2e^{j\pi/2}e^{-j\pi 6t} \\
z(t) &= x(t)y(t) = e^{j\pi/2}e^{j\pi 14t} + e^{-j\pi/2}e^{-j\pi 14t} + e^{-j\pi/2}e^{j\pi 2t} + e^{j\pi/2}e^{-j\pi 2t}
\end{aligned}
$$

(c) These plots can be made using the answers from part (b). Make sure that you put labels at the axis (so f in Hz, or rad/s). Also, you need to specify for each frequency component the corresponding phase angle.

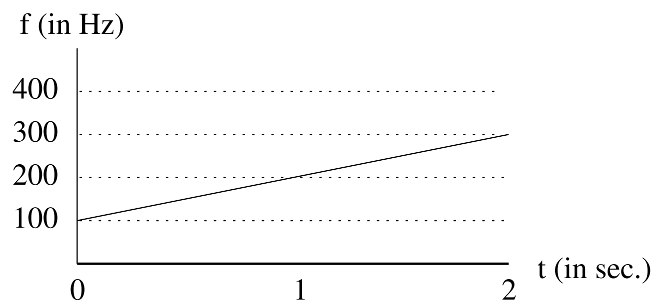


(d) A chirp signal is of the form $x(t) = A\cos(2\pi\alpha t^2 + 2\pi\beta t + \phi)$. At $t = 0$, the phase is zero, and the deflection is 2, so we find $\phi = 0$, and $A = 2$. The instantaneous frequency in Hz is the derivative of the angle function divided by 2π, i.e. $f_i = \frac{d}{dt}(\alpha t^2 + \beta t) = 2\alpha t + \beta$. Since $f_0 = 220$, we find $220 = 0\alpha + \beta = \beta = 220$. Since $f_3 = 2320$ and $\beta = 220$, we find $2320 = 6\alpha + 220$. Hence, $\alpha = 350$. In conclusion, we find $x(t) = 2\cos(2\pi 350t^2 + 2\pi 220t)$.

Problem 2: Instantaneous frequency, spectrograms, and sampling
Parts (a), (b), and (c) are worth 4 points. Part (d) is worth 2 points.

(a) $f_i(t) = \frac{1}{2\pi}\frac{d}{dt}(200\pi t + 100\pi t^2) = 100t + 100$.

(b) The spectrogram is simply the straight line $f_i(t)$ from part (a).



(c) We find the discrete frequencies $\hat{\omega}_0 = \pi/2$ and $\hat{\omega}_1 = \pi/4$:

$$y[n] = y(n \cdot T_s) = 6\cos(\frac{30\pi n}{60} + \pi/2) + \cos(\frac{15\pi n}{60} + \pi/2) = 6\cos(n\frac{\pi}{2} + \pi/2) + \cos(n\frac{\pi}{4} + \pi/2)$$

(d) The signal $y(t)$ is frequency-limited, with the highest frequency being 15Hz. So, a sampling frequency greater than 30Hz (Nyquist freq.) is sufficient to avoid aliasing. Since the sampling frequency is 60Hz, no aliasing will occur.

Problem 3: Fourier analysis Parts (a), (c), and (d) are worth 4 points. Part (b) is worth 8 points.

(a) Since $T_0 = 1/100$, we have the base frequency $f = 100$Hz. Therefore, using the Fourier synthesis formula, we find $x(t) = 3 + 2\cos(2\pi 100t - \pi/2) + 4\cos(2\pi 500t)$.
Hence, $DC = 3$, $A = 2$, $f_0 = 100$, $\phi_0 = -\pi/2$, $B = 4$, $f_1 = 500$, and $\phi_1 = 0$.

(b) According to the Fourier analysis formula we find:

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt = \frac{1}{5}\left(\int_0^1 e^{-j2\pi kt/5}\,dt + \int_4^5 e^{-j2\pi kt/5}\,dt\right)$$

For the DC -term (i.e. $k = 0$, so $e^{-j2\pi kt/5} = e^0 = 1$) this reduces to:

$$a_0 = \frac{1}{5}\left(\int_0^1 1\,dt + \int_4^5 1\,dt\right) = \frac{1}{5}(1 - 0 + 5 - 4) = \frac{2}{5}$$

For other k we find:

$$\begin{aligned} a_k &= \frac{1}{5}\left(\left[\frac{e^{-j2\pi kt/5}}{-j2\pi k/5}\right]_{t=0}^{t=1} + \left[\frac{e^{-j2\pi kt/5}}{-j2\pi k/5}\right]_{t=4}^{t=5}\right) = \frac{j}{2\pi k}\left(\left[e^{-j2\pi kt/5}\right]_{t=0}^{t=1} + \left[e^{-j2\pi kt/5}\right]_{t=4}^{t=5}\right) \\ &= \frac{j}{2\pi k}\left(e^{-j2\pi k/5} - 1 + e^{-j2\pi k} - e^{-j8\pi k/5}\right) \\ &= \frac{j}{2\pi k}\left(e^{-j2\pi k/5} - e^{-j8\pi k/5}\right) \end{aligned}$$

Note that we used $e^{-j2\pi k} = 1$ (for integer k) in the last step of this derivation.

(c) We can use the answer from (b), since $y(t)$ is simply the signal shifted by 1 second, i.e. a phase shift of $-2\pi/5$. So, we simply find $b_k = e^{-j2\pi/5}a_k = \frac{j}{2\pi k}\left(e^{-j4\pi k/5} - 1\right)$ for all k.

(d) The signal is an AM-signal, which we first need to convert into a sum of cosines:

$$\begin{aligned} z(t) &= 5 + 2\sin(2\pi 120t)\cos(2\pi 30t) = 5 + \sin(2\pi 150t) + \sin(2\pi 90t) \\ &= 5 + \cos(2\pi 150t - \pi/2) + \cos(2\pi 90t - \pi/2) \end{aligned}$$

Now, we can read the Fourier coefficients directly from the formula. However, we first need to determine the fundamental frequency $f_0 = \gcd(150, 90) = 30$Hz. So, the cases are $k = 0$, $k = \pm 3$ and $k = \pm 5$.

$$a_k = \begin{cases} \frac{1}{2}e^{j\pi/2} & \text{for } k = -5 \\ \frac{1}{2}e^{j\pi/2} & \text{for } k = -3 \\ 5 & \text{for } k = 0 \text{ (DC-term)} \\ \frac{1}{2}e^{-j\pi/2} & \text{for } k = 3 \\ \frac{1}{2}e^{-j\pi/2} & \text{for } k = 5 \\ 0 & \text{for all other } k \end{cases}$$

Problem 4: LTI-systems

Each part is worth 5 points.

(a) Causal: the output at location n is only dependent on the input at the locations $n-k$, where $k \geq 0$. Clearly, this is true for y_0, while it is not for y_1 due to the index-flip at $n = 0$.
Linear: For a linear system F we have $F(\alpha x_0[n] + \beta x_1[n]) = \alpha F(x_0[n]) + \beta F(x_1[n])$ for all n. This is clearly not true for y_0 since $(\alpha x[n-1])^2 \neq \alpha(x[n-1])^2$. The system y_1 is clearly linear.
Time invariant: shifting the output by a delay d results in the same signal as shifting the input by the same delay, and compute the corresponding output. Clearly y_0 is time invariant, but y_1 is not due to the index-flip at $n = 0$. In conlusion:

system	causal	linear	time invariant
$y_0[n] = (x[n-1])^2$	Yes	No	Yes
$y_1[n] = 2x[-n]$	No	Yes	No

(b) The output is the convolution y[n] = $[1, 2, 3, 2, 1] * [1, 1, 1, 1, 1, 1, 1, 1, ...] = [1, 3, 6, 8, 9, 9, 9, 9, 9, ...]$, where $y[n] = 0$ for $n < 0$.

(c) The output of system h_1 will be the convolution $[1, 1, 1] * [1, 0, 1] = [1, 1, 2, 1, 1]$. So, we search for a system h_2 such that $h_2 * [1, 1, 2, 1, 1] = [1, 2, 4, 4, 4, 2, 1] = y[n]$. The signal y has 7 samples, while $[1, 1, 2, 1, 1]$ has 5 samples. Therefore, h_2 has three samples, let us say $h_2 = [a, b, c]$. We compute

$$[a, b, c] * [1, 1, 2, 1, 1] = [a, a+b, 2a+b+c, a+2b+c, a+b+2c, b+c, c] = [1, 2, 4, 4, 4, 2, 1]$$

It is clear that $a = b = c = 1$, so we find $h_2 = [1, 1, 1]$.

(d) Again, the output of system h_1 will be the convolution $[1, 1, 1] * [1, 0, 1] = [1, 1, 2, 1, 1]$. So, we search for a system h_2 such that $h_2 * x = h_2 * [1, 0, 1] = y - [1, 1, 2, 1, 1] = [0, 2, 1, 2, 1]$. We conclude that h_2 has 3 samples, i.e. $h_2 = [a, b, c]$. We compute

$$[a, b, c] * [1, 0, 1] = [a, b, a+c, b, c] = [0, 2, 1, 2, 1]$$

We find $a = 0$, $b = 2$, and $c = 1$. So, we find $h_2 = [0, 2, 1]$.

Problem 5: frequency responses and z-transforms
Each part is worth 5 points.

(a) The system $y[n] = \frac{1}{5}(x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]$ has the following frequency response $H(e^{j\omega})$ and system function $H(z)$:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{5} + \frac{1}{5}e^{-j\hat{\omega}} + \frac{1}{5}e^{-j2\hat{\omega}} + \frac{1}{5}e^{-j3\hat{\omega}} + \frac{1}{5}e^{-j4\hat{\omega}} \\ H(z) &= \frac{1}{5} + \frac{1}{5}z^{-1} + \frac{1}{5}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{5}z^{-4} \end{aligned}$$

(b) Using $\cos\theta = (e^{j\theta} + e^{-j\theta})/2$, we find

$$H(z) = 1 - 2\cos(\hat{\omega})z^{-1} + z^{-2} = 1 - (e^{j\hat{\omega}} + e^{-j\hat{\omega}})z^{-1} + z^{-2} = (1 - e^{j\hat{\omega}}z^{-1})(1 - e^{-j\hat{\omega}}z^{-1})$$

We want $H(z) = 0$, so the roots are $z_0 = e^{j\hat{\omega}}$ and $z_1 = e^{-j\hat{\omega}}$. The conclusion is that the frequencies $\pm\hat{\omega}$ are completely removed by this system. So, if we feed this system $x[n] = 1 + 3\sin(n\hat{\omega})$, then only the DC-term will 'survive', i.e. $y[n] = [...., 2 - 2\cos(\hat{\omega}), 2 - 2\cos(\hat{\omega}), 2 - 2\cos(\hat{\omega}), ...]$.

(c) The signal $x[n]$ first needs to be written as a sum:

$$x[n] = 1 + \cos(\frac{\pi n}{3})\cos(\frac{\pi n}{4}) = 1 + \frac{1}{2}\left(\cos\left(\frac{\pi n}{12}\right) + \cos\left(\frac{7\pi n}{12}\right)\right)$$

The DC-term is removed by $h_1 = [1, -1]$. Obviously, we can remove the two cosine-terms using a 24-point averager, so we find the system

$$\begin{aligned} h[n] &= [1,-1] * [1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1] \\ &= [1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,-1] \\ &= \delta[n] - \delta[n-24] \end{aligned}$$

The system function is therefore: $H(z) = 1 - z^{-24}$
The corresponding difference equation: $y[n] = x[n] - x[n-24]$

[Note: we can find a more specific solution (but this was not asked for), by using the result of part (b). The two cosine-terms are removed by $h_2 = [1, \alpha, 1]$ and $h_3 = [1, \beta, 1]$ where $\alpha = -2\cos(\pi/12)$ and $\beta = -2\cos(7\pi/12)$. So, the system is $h = h1 * h_2 * h_3 = [1, \alpha+\beta-1, 2+\alpha\beta-\alpha-\beta, \alpha+\beta-2\alpha\beta, 1-\alpha-\beta, -1]$. This yields much more complicated expressions.]

(d) Assume that H_2 is the inverse of H_1, and H_2 is a FIR filter. Then $H_2(z)H_1(z) = 1$, which means that $H_2(z) = \frac{1}{H_1(z)}$. However, if $H_1(z)$ is a polynomial in z^{-1}, then $\frac{1}{H_1(z)}$ can not be a polynomial in z^{-1}, which is a contrasdiction with our assumption. We conclude that H_2 cannot be a FIR filter.